Posterior Inference in Gaussian Processes is an Affine Transformation

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A Gaussian

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} \right) \tag{1}$$

The Conditional / Posterior Distribution

$$f_1 \mid \left(f_2 = \hat{f}\right) \sim \mathcal{N}\left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\hat{f} - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^{T}\right) \tag{2}$$

(Standard result, found in any good statistics / ML textbook).

Affine Transformations of Gaussians

Gaussians are closed under affine transforms; the affine transform of a Gaussian RV is another Gaussian RV.

$$f \sim \mathcal{N}(\mu, \Sigma) \tag{3}$$

$$Pf + q \sim \mathcal{N}\left(P\mu + q, P\Sigma P^T\right) \tag{4}$$

(Standard result, found in any good statistics / ML textbook).

Same thing?

- Question: Can an affine transformation be found which transforms the distribution of $[f_1, f_2]^T$ into that of $f_1 \mid \left(f_2 = \hat{f}\right)$?

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- Question: Can an affine transformation be found which transforms the distribution of $[f_1, f_2]^T$ into that of $f_1 \mid \left(f_2 = \hat{f}\right)$?
- Answer: Yes. $P := \left[\mathcal{I}, -\Sigma_{12}\Sigma_{22}^{-1}\right]$, $q := \Sigma_{12}\Sigma_{22}^{-1}\hat{f}$.

Same thing?

$$\left[\mathcal{I}, -\Sigma_{12}\Sigma_{22}^{-1}\right]\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} + \Sigma_{12}\Sigma_{22}^{-1}\hat{f} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}\left(\hat{f} - \mu_2\right) \tag{5}$$

$$\begin{bmatrix}\mathcal{I}\\ -\Sigma_{22}^{-1}\Sigma_{12}^{T}\end{bmatrix}^{T}\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{12}^{T} & \Sigma_{22}\end{bmatrix}\begin{bmatrix}\mathcal{I}\\ -\Sigma_{22}^{-1}\Sigma_{12}^{T}\end{bmatrix} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^{T}. \tag{6}$$

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- Gaussians are closed under affine transformations.
- Posterior inference in a Gaussian (process) RV *is* an affine transformation of the prior.